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FOREWORD

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PROPAGATION OF ELASTIC WAVES IN TWO-COMPONENT MEDIA
USSR

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To date a number of works have been devoted to the dynamics of two-component media; for example, Ya. I. Frenkel' [1], Kh. A. Rakhmatulin [2], Biot [3,4], and Zwicker and Costain [5]. However, the basic problem regarding the construction of equations of motion of two-component media has still not ultimately been solved but requires additional study and experimental checking. Below we shall examine the simplest case of movement -- propagation of elastic waves in a homogeneous isotropic medium, consisting of solid and liquid components. The problems of the reflection of plastic waves and of surface waves on the free boundary of a semi-space were solved. It was shown that the relationship between stresses and deformations established by Ya. I. Frenkel' is equivalent to a similar relationship established by M. A. Biot, while the equations of motion of the latter are the more common.

1. Basic equations. We can consider the interpenetrating motion of solid and liquid components to be the motion of a liquid in a deformable porous medium. We shall assume that the pore dimensions are small compared to the distance in which the kinematic and dynamic characteristics of motion undergo substantial change. This permits us to regard both media as being continuous throughout, and at each point in space we shall thus have two vectors of displacement: the displacement vector u of the solid phase (of the skeleton of the porous medium) and the displacement vector v of the liquid. As shown by Ya. I. Frenkel' (1), the total tensor of stresses in the skeleton (with pressure of the liquid present in the pores) can be written in the form

$$P_{ik} = L \theta \delta_{ik} + 2G u_{ik} + R_0 \left(1 - m - \frac{K}{K_0}\right) \varphi \delta_{ik} \quad (1.1)$$

$$K = L + 2/3 G, \quad u_{ik} = 1/2 \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)$$

$$\theta = u_{11} + u_{22} + u_{33} = \text{div } u, \quad \varphi = - \frac{\Delta p}{p}$$

where L , G and K are respectively the Lamé coefficients and the modulus of omnilateral compression of the porous skeleton with empty pores, K_0 is the true modulus of compressibility of the solid phase, R_0 the modulus of compressibility of the liquid, m the porosity, u_{ik} the tensor of skeleton deformation, θ the relative change in skeleton volume, φ the relative change in true density of the liquid, and δ_{ik} Kronecker's symbol, equal to unity when $i = k$, and to zero when $i \neq k$.

The change in porosity, according to (1), can be expressed by the formula

$$\Delta m = a \left(\theta - \frac{R_0}{K_0} \varphi \right) \quad (1.2)$$

The constant coefficient a is some additional parameter, characterizing the elastic properties of the porous medium (with porosity constant, $a = 0$).

A small change in pressure in the liquid is associated with a small change in density by the equality

$$\Delta p = - R_0 \varphi \quad (1.3)$$

The discontinuity equation, expressing the law of the preservation of matter, has the form

$$\frac{\partial}{\partial t} (mp) + \text{div} \left(mp \frac{\partial v}{\partial t} \right) = 0 \quad (1.4)$$

Linearizing it and proceeding to finite differences, we obtain

$$m \Delta p + p \Delta m + m p \varepsilon = 0 \quad \varepsilon = \operatorname{div} v \quad (1.5)$$

where ε is the relative change in the volume of the liquid. Taking into account (1.2), we find

$$\varphi = \beta \left(\frac{a}{m} \theta + \varepsilon \right), \quad \beta = \frac{1}{1 + a R_0 / m K_0} \quad (1.6)$$

Substituting (1.6) in (1.1) and (1.3), we reduce them to the form

$$P_{ik} = \lambda \theta \delta_{ik} + 2 \mu u_{ik} + Q \varepsilon \delta_{ik}, \quad (1.7)$$

$$\sigma = -m \Delta p = Q' \theta + R \varepsilon$$

where σ is the force acting on the liquid in terms of a unit cross section of the porous medium, and introduce the symbols

$$\lambda = L + \beta \frac{a}{m} \left(1 - m - \frac{K}{K_0} \right) R_0, \quad \mu = G$$

$$Q = \beta \left(1 - m - \frac{K}{K_0} \right) R_0, \quad Q' = \beta a R_0, \quad R = \beta m R_0 \quad (1.8)$$

For the elementary operation of forces acting on the solid and liquid component, we have

$$d'A = P_{ik} du_{ik} + \sigma d\varepsilon = \lambda \theta d\theta + 2 \mu u_{ik} du_{ik} + Q \varepsilon d\theta + Q' \theta d\varepsilon + R \varepsilon d\varepsilon$$

For the existence of potential energy, this expression should be a complete differential; consequently, the equality $Q = Q'$ ensues and, hence, taking into account (1.8) we obtain

$$a = 1 - m - \frac{K}{K_0} \quad (1.9)$$

Equations (1.7) coincide with the relationships between stresses and deformations established by Biot (3). If shear of the skeleton is disregarded, only the diagonal members will remain in the tensor of stresses P_{ik} , i.e., $P_{ik} = -P_1 \delta_{ik}$, where P_1 is the force of the pressure on the skeleton in terms of a unit cross section of the porous medium. Hence, equation (1.7) takes the form:

$$-P_1 = (\lambda + 2/3 \mu) \theta + Q \varepsilon, \quad -m \Delta p = Q \theta + R \varepsilon \quad (1.10)$$

Let $R_0 \ll K \ll K_0$ (this can occur in the case when the pores are filled with gas). According to (1.8) and (1.2), we obtain for this case

$$\lambda + \frac{2}{3} \mu = K, \quad R = m R_0, \quad Q = (1 - m) R_0, \quad \Delta m = (1 - m) \theta \quad (1.11)$$

Let us introduce the quantity

$$P_2 = m p - m_0 p_0 \quad (1.12)$$

Here m_0 and p_0 are equilibrium values of porosity and the pressure of the liquid (of the gas). Making use of the equality

$$\frac{\partial P_2}{\partial t} = m \frac{\partial p}{\partial t} + p_0 \frac{\partial m}{\partial t},$$

and considering (1.10) and (1.11), we obtain

$$-\frac{\partial P_1}{\partial t} = K \frac{\partial \theta}{\partial t} - \frac{1 - m}{m} \frac{\partial P_2}{\partial t}, \quad -\frac{\partial P_2}{\partial t} = m R_0 \frac{\partial \varepsilon}{\partial t} + (1 - m)(R_0 - p_0) \frac{\partial \theta}{\partial t}$$

which coincides with the equations of Zwicker and Costain (5).

Let us now obtain the equations of motion, disregarding at first the viscosity of the liquid.

The force acting on the skeleton (in a unit volume of the porous medium) is equal to the derivative in time of the total impulse of the skeleton (in this volume). The latter is made up of the impulse of the skeleton with absolutely empty pores, and an additional impulse arising from the relative motion of the components of the medium, i.e.,

$$j_1 = p_1 \frac{\partial u}{\partial t} + p_{12} \left(\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t} \right) = p_{11} \frac{\partial u}{\partial t} + p_{12} \frac{\partial v}{\partial t} \quad (1.14)$$

Thus, we have

$$p_{11} \frac{\partial^2 u_1}{\partial t^2} + p_{12} \frac{\partial^2 v_1}{\partial t^2} = \frac{\partial p_{1k}}{\partial x_k} \quad (1.15)$$

Similarly, we find the total impulse of the liquid

$$j_2 = p_2 \frac{\partial v}{\partial t} + p_{12} \left(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t} \right) = p_{12} \frac{\partial u}{\partial t} + p_{22} \frac{\partial v}{\partial t} \quad (1.16)$$

and obtain the second equation of motion:

$$p_{12} \frac{\partial^2 u_1}{\partial t^2} + p_{22} \frac{\partial^2 v_1}{\partial t^2} = \frac{\partial \sigma}{\partial x_1} \quad (1.17)$$

In accordance with (4), we shall call the coefficient p_{12} the coefficient of dynamic coupling between the skeleton and the liquid; p_1 and p_2 are the masses of solid and liquid components in a unit volume of medium, and p_{11} and p_{12} can be regarded as certain "effective masses" of these components:

$$p_{11} = p_1 - p_{12}, \quad p_{22} = p_2 - p_{12} \quad (1.18)$$

Since during the motion of a solid body in a liquid its "effective mass" is greater than its true mass ($p_{11} > p_1$), the coefficient p_{12} must be negative.

To take into account viscosity in equations (1.15) and (1.17) it is necessary to add a member proportional to the difference in the velocities of the two components:

$$p_{11} \frac{\partial^2 u_1}{\partial t^2} + p_{12} \frac{\partial^2 v_1}{\partial t^2} + b \left(\frac{\partial v_1}{\partial t} - \frac{\partial u_1}{\partial t} \right) = \frac{\partial p_{1k}}{\partial x_k} \quad (1.19)$$

$$p_{12} \frac{\partial^2 u_1}{\partial t^2} + p_{22} \frac{\partial^2 v_1}{\partial t^2} + b \left(\frac{\partial v_1}{\partial t} - \frac{\partial u_1}{\partial t} \right) = -m \frac{\partial p}{\partial x_1}$$

The constant b can be determined from conditions satisfying Darcy's (Darsi) law for the particular case of the steady-state flow of a liquid. Thus, we find

$$b = \frac{\mu m^2}{k} \quad (1.20)$$

where μ is the coefficient of viscosity, and k the coefficient of permeability, proportional to the porosity and the square of the pore diameter:

$$k = \text{const } m \delta^2 \quad (1.21)$$

If we disregard the coefficient of dynamic coupling p_{12} in (1.19), we shall obtain the equation of motion derived by Ya. I. Frenkel' (1). In the case of harmonic waves with a frequency ω (in which the dependence of displacement vectors on time can be expressed by the factor $e^{-i\omega t}$) equation (1.19) can be written also in the form

$$p_1 \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial p_{ik}}{\partial x_k} / s \left(\frac{\partial v_i}{\partial t} - \frac{\partial u_i}{\partial t} \right), \quad p_2 \frac{\partial^2 v_i}{\partial t^2} = -m \frac{\partial p}{\partial x_i} / s \left(\frac{\partial u_i}{\partial t} - \frac{\partial v_i}{\partial t} \right) \quad (1.22)$$

where

$$s = b - i\omega p_{12} \quad (1.23)$$

If we disregard shear of the skeleton, i.e.,

$$p_{ik} = -p_1 \delta_{ik} = -(1-m) p_s \delta_{ik}$$

(p_s - true pressure in the skeleton), from (1.22) we obtain

$$\frac{\partial^2 u_i}{\partial t^2} = -\frac{1}{p_s} \frac{\partial p_s}{\partial x_i} / K_{12} \left(\frac{\partial v_i}{\partial t} - \frac{\partial u_i}{\partial t} \right), \quad \frac{\partial^2 v_i}{\partial t^2} = -\frac{1}{p} \frac{\partial p}{\partial x_i} / K_{21} \left(\frac{\partial u_i}{\partial t} - \frac{\partial v_i}{\partial t} \right) \quad (1.24)$$

Here p_s and p are the true densities of the solid and liquid components:

$$p_s = \frac{p_1}{1 - m}, \quad p = \frac{p_2}{m}, \quad K_{12} = \frac{s}{p_1}, \quad K_{21} = \frac{s}{p_2} \quad (1.25)$$

Equations (1.24) coincide with the equations of motion derived by Kh. A. Rakhmatulin [2] for multi-component liquid and gaseous media.

Let us resolve the displacement vectors into laminar and solenoidal components

$$\begin{aligned} u &= u_1 + u_t, & \text{rot } u_1 &= 0, & \text{div } u_t &= 0 \\ v &= v_1 + v_t, & \text{rot } v_1 &= 0, & \text{div } v_t &= 0 \end{aligned} \quad (1.26)$$

Equations (1.19), taking into account (1.7), (1.9) and (1.26), are reduced to the following system:

$$\begin{aligned} p_{11} \frac{\partial^2 u_1}{\partial t^2} + p_{12} \frac{\partial^2 v_1}{\partial t^2} + b \frac{\partial}{\partial t} (u_1 - v_1) &= (\lambda + 2\mu) \nabla^2 u_1 + Q \nabla^2 v_1 \\ p_{12} \frac{\partial^2 u_1}{\partial t^2} + p_{22} \frac{\partial^2 v_1}{\partial t^2} + b \frac{\partial}{\partial t} (v_1 - u_1) &= Q \nabla^2 u_1 + R \nabla^2 v_1 \\ p_{11} \frac{\partial^2 u_t}{\partial t^2} + p_{12} \frac{\partial^2 v_t}{\partial t^2} + b \frac{\partial}{\partial t} (u_t - v_t) &= \mu \nabla^2 u_t \\ p_{12} \frac{\partial^2 u_t}{\partial t^2} + p_{22} \frac{\partial^2 v_t}{\partial t^2} + b \frac{\partial}{\partial t} (v_t - u_t) &= 0 \end{aligned} \quad (1.27)$$

In the case of monochromatic waves with a frequency ω , the first two equations (1.27), by means of linear transformation

$$u_1 = u_1 + u_2, \quad v_1 = M_1 u_1 + M_2 u_2 \quad (1.28)$$

and after insertion of symbols,

$$\gamma_{11} = \frac{p_{11}}{p}, \quad \gamma_{12} = \frac{p_{12}}{p}, \quad \gamma_{22} = \frac{p_{22}}{p}, \quad p = p_{11} + p_{22} + 2p_{12} \quad (1.29)$$

$$\sigma_{11} = \frac{\lambda + 2\mu}{H}, \quad \sigma_{12} = \frac{Q}{H}, \quad \sigma_{22} = \frac{R}{H}, \quad H = \lambda + 2\mu + R + 2Q$$

$$c^2 = \frac{H}{p}$$

reduce to the form

$$\nabla^2 u_1 + k_1^2 u_1 = 0, \quad \nabla^2 u_2 + k_2^2 u_2 = 0 \quad (1.30)$$

where

$$k_1^2 = z_1 \left(\frac{\omega}{c}\right)^2, \quad k_2^2 = z_2 \left(\frac{\omega}{c}\right)^2 \quad (1.31)$$

$$(\sigma_{11}\sigma_{22} - \sigma_{12}^2)z^2 - (\sigma_{11}\gamma_{22} + \sigma_{22}\gamma_{11} - 2\sigma_{12}\gamma_{12})z +$$

$$+ \gamma_{11}\gamma_{22} - \gamma_{12}^2 - \frac{1b}{\omega p} (z - 1) = 0 \quad (1.32)$$

while the transformation coefficients (1.28) are determined by the formulas

$$M_1 = \frac{-\gamma_{12} + z_1\sigma_{12} + i\gamma}{\gamma_{22} - z_1\sigma_{22} + i\gamma}, \quad M_2 = \frac{-\gamma_{12} + z_2\sigma_{12} + i\gamma}{\gamma_{22} - z_2\sigma_{22} + i\gamma}, \quad \gamma = \frac{b}{p\omega} \quad (1.33)$$

Equations (1.30) describe the propagation of longitudinal waves of types I and II.

If the coupling between the solid and liquid components of the medium is weak

$$\gamma_{12} = 0, \quad \sigma_{12} = 0, \quad \gamma = 0,$$

then from (1.32) we find

$$z_1 = \frac{\gamma_{11}}{\sigma_{11}}, \quad z_2 = \frac{\gamma_{22}}{\sigma_{22}} \quad (1.34)$$

hence, it is obvious that the velocities of I and II longitudinal waves are equal to the velocities of waves propagating separately in continuous elastic and continuous liquid media. According to (1.33) we have in this case

$$M_1 = 0, \quad M_2 = \infty \quad (1.35)$$

Let us find the approximated values for the roots of equations (1.32) under the condition $\gamma \gg 1$, which corresponds to the case of low frequency or high viscosity:

$$z_1 = 1 - \frac{1}{\gamma} (\sigma_{11}\sigma_{22} - \sigma_{12}^2 - \sigma_{11}\gamma_{22} - \sigma_{22}\gamma_{11} + 2\sigma_{12}\gamma_{12} + \gamma_{11}\gamma_{22} - \gamma_{12}^2)$$

$$z_2 = 1 \frac{\gamma}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \quad (1.36)$$

As easily seen, moreover, the damping coefficients of the waves (being the imaginary values k_1 and k_2) will be proportional: for I longitudinal -- to the square of the frequency, and for II longitudinal -- to the square root of the frequency. Therefore, at very low frequency, longitudinal wave I experiences insignificant attenuation, while longitudinal wave II, because of great attenuation, practically disappears.

The second two equations (1.27), describing the propagation of the transverse wave, is reduced to the form

$$v_t = M_t u_t, \quad \nabla^2 u_t + k_t^2 u_t = 0 \quad (1.37)$$

where

$$M_t = \frac{-\gamma_{12} + i\gamma}{\gamma_{22} + i\gamma}, \quad k_t^2 = \frac{p_1 + p_2 M_t}{\mu} \omega^2 \quad (1.38)$$

At low frequency, here as for longitudinal wave I, the damping coefficient is proportional to the square of the frequency.

The existence of three types of waves in porous media, as well as the nature of their attenuation, was established in works [1,4]. If $\gamma \gg 1$, the effect of viscosity can be disregarded. However, at the same time it should be borne in mind that ω must remain smaller than the frequency at which the wave becomes comparable to the dimensions of the pores.

2. Reflection of plane waves on the free boundary of a semi-space. Boundary conditions on the free surface of the medium will be

$$P_{yy} = 0, \quad P_{xy} = 0, \quad \sigma = 0 \quad (2.1)$$

or on the basis of (1.7)

$$\lambda \theta + 2\mu u_{yy} + Q\varepsilon = 0$$

$$u_{xy} = 0 \quad (2.2)$$

$$Q\theta + R\varepsilon = 0$$

Introducing the potentials of the longitudinal and transverse waves, and recalling (1.26), (1.28) and (1.37), we shall write expressions for the total vectors of displacement of the skeleton and the liquid

$$u = \text{grad } \varphi_1 + \text{grad } \varphi_2 + \text{rot } \Psi, \quad v = M_1 \text{grad } \varphi_1 + M_2 \text{grad } \varphi_2 + M_t \text{rot } \Psi \quad (2.3)$$

after which the boundary conditions reduce to the form

$$(\lambda + QM_1) \nabla^2 \varphi_1 + (\lambda + QM_2) \nabla^2 \varphi_2 + 2\mu \left(\frac{\partial^2 \varphi_1}{\partial y^2} + \frac{\partial^2 \varphi_2}{\partial y^2} + \frac{\partial^2 \Psi}{\partial x \partial y} \right) = 0$$

$$2 \left(\frac{\partial^2 \varphi_1}{\partial x \partial y} + \frac{\partial^2 \varphi_2}{\partial x \partial y} \right) + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = 0 \quad (2.4)$$

$$(Q + RM_1) \nabla^2 \varphi_1 + (Q + RM_2) \nabla^2 \varphi_2 = 0$$

Let us examine the case of the incidence of longitudinal wave I on the boundary. Three boundary conditions can be satisfied, simply assuming that the incident wave sets up at the boundary a system of three reflected waves: two longitudinal and one transverse. Thus, the total sound field in the medium will be (see figure)

$$\varphi_1 = A e^{ik_1} (x \sin \theta_1 + y \cos \theta_1) + A W_1 e^{ik_1} (x \sin \theta_1 - y \cos \theta_1)$$

$$\varphi_2 = A W_2 e^{ik_2} (x \sin \theta_2 - y \cos \theta_2) \quad (2.5)$$

$$\psi = A W_t e^{ik_t} (x \sin \theta_t - y \cos \theta_t)$$

where W_1 , W_2 and W_t are the unknown reflection coefficients (for brevity, we shall omit the factor $e^{-i\omega t}$).

Substituting (2.5) in (2.4) and solving the system of linear equations obtained, we find

$$W_1 = - \frac{F_1 - L_{12} F_2 - \mu (\sin 2\theta_1 + L_{12} \sin 2\theta_2) \operatorname{tg} 2\theta_t}{F_1 - L_{12} F_2 + (\sin 2\theta_1 - L_{12} \sin 2\theta_2) \operatorname{tg} 2\theta_t},$$

$$W_2 = - \left(\frac{k_1}{k_2} \right)^2 L_{12} (1 + W_1) \quad (2.6)$$

$$W_t = \frac{1}{\cos 2\theta_t} \left(\frac{k_1}{k_t} \right)^2 [\sin 2\theta_1 + L_{12} \sin 2\theta_2 - W_1 (\sin 2\theta_1 - L_{12} \sin 2\theta_2)]$$

Here we insert the symbols

$$\begin{aligned} F_1 &= K_1 - 2\mu \sin^2 \theta_1, & F_2 &= K_2 - 2\mu \sin^2 \theta_2, \\ K_1 &= \lambda + 2\mu / QM_1, & K_2 &= \lambda + 2\mu / QM_2, & L_{12} &= \frac{Q / RM_1}{Q / RM_2} \end{aligned} \quad (2.7)$$

At low frequencies, the reflection coefficient W_2 will, according to (1.36) and (1.31), be proportional to ω . Therefore, in the first approximation we can consider that only type I longitudinal and transverse waves will be reflected from the boundary.

In the case of normal incidence of the wave on the boundary ($\theta_1 = \theta_2 = \theta_t = 0$)

$$W_1 = -1, \quad W_2 = 0, \quad W_t = 0 \quad (2.8)$$

i.e., only longitudinal wave I will be reflected.

Considering the porosity to be tending toward zero and at the same time that conditions (1.35) prevail, and also the equalities

$$\lambda + 2\mu = p_s c_l^2, \quad \mu = p_s c_t^2, \quad \frac{k_l}{k_t} = \frac{c_t}{c_l} = \frac{\sin \theta_t}{\sin \theta_1}$$

where c_l and c_t are the velocities of the longitudinal and transverse waves, we obtain the well-known reflection coefficients for a continuous isotropic elastic medium [6]:

$$\begin{aligned} W_t &= \frac{c_t \cos \theta_1 \tan^2 2\theta_t - c_l \cos \theta_t}{c_t \cos \theta_1 \tan^2 2\theta_t + c_l \cos \theta_t}, & W_2 &= 0 \\ W_1 &= \frac{c_t \sin 2\theta_1 - 2c_t \cos \theta_t}{c_l \cos 2\theta_t + c_t \cos \theta_1 \tan^2 2\theta_t + c_l \cos \theta_t} \end{aligned} \quad (2.9)$$

Let a type II longitudinal wave be incident on the boundary. Similar to the preceding, we find

$$W_1 = -\frac{1}{L_{12}} \left(\frac{k_2}{k_1} \right)^2 (1 + W_2)$$

$$W_2 = - \frac{F_1 - L_{12}F_2 + \mu (\sin^2 \theta_1 + L_{12}\sin^2 \theta_2) \operatorname{tg} 2\theta_t}{F_1 - L_{12}F_2 + \mu (\sin^2 \theta_1 - L_{12}\sin^2 \theta_2) \operatorname{tg} 2\theta_t} \quad (2.10)$$

$$W_t = \frac{1}{L_{12}\cos 2\theta_t} \left(\frac{k_2}{k_t}\right)^2 [\sin^2 \theta_1 + L_{12}\sin^2 \theta_2 + W_2 (\sin^2 \theta_1 - L_{12}\sin^2 \theta_2)]$$

For the case of a transverse wave incident on the boundary we have

$$W_1 = - \left(\frac{k_t}{k_1}\right)^2 \frac{\mu \sin^2 \theta_t}{F_1 - L_{12}F_2} (1 - W_t), \quad W_2 = L_{12} \left(\frac{k_t}{k_2}\right)^2 \frac{\mu \sin^2 \theta_t}{F_1 - L_{12}F_2} (1 - W_t) \quad (2.11)$$

$$W_t = - \frac{F_1 - L_{12}F_2 - \mu (\sin^2 \theta_1 - L_{12}\sin^2 \theta_2) \operatorname{tg} 2\theta_t}{F - L F + \mu (\sin^2 \theta_1 - L_{12}\sin^2 \theta_2) \operatorname{tg} 2\theta_t}$$

In the limiting case of disappearing porosity, from (2.11) we obtain

$$W_1 = - \frac{2c_1 \cos \theta_t \operatorname{tg} 2\theta_t}{c_t \cos \theta_1 \operatorname{tg}^2 2\theta_t + c_1 \cos \theta_t}, \quad W_2 = 0, \quad W_t = \frac{c_t \cos \theta_1 \operatorname{tg}^2 2\theta_t - c_1 \cos \theta_t}{c_t \cos \theta_1 \operatorname{tg}^2 2\theta_t + c_1 \cos \theta_t} \quad (2.12)$$

which coincides with the well-known expressions for reflection coefficients in continuous elastic media (6).

To conclude this section, we shall note the case of total internal reflection.

We shall neglect viscosity and consider that the velocities of the two longitudinal and transverse waves satisfy the inequalities

$$c_1 > c_2 > c_t \quad (2.13)$$

If the transverse wave is incident on the boundary, the case of total internal reflection will occur when

$$\sin \theta_t > \frac{c_t}{c_2} \quad (2.14)$$

The angles θ_1 and θ_2 , determined by the relationship

$$\sin \theta_1 = \frac{c_1}{c_t} \sin \theta_t, \quad \sin \theta_2 = \frac{c_2}{c_t} \sin \theta_t \quad (2.15)$$

prove meanwhile to be complex, while type I longitudinal and transverse waves are nonuniform, their amplitudes decreasing exponentially with distance from the boundary. In the case of the type II longitudinal wave incident on the boundary at the angle θ_2 , satisfying the condition

$$\sin \theta_2 > \frac{c_2}{c_1} \quad (2.16)$$

the angle θ_1 will be complex, and longitudinal wave I nonuniform.

3. Surface waves. Let us examine surface waves as a degenerated case of the reflection of surface waves. For this, as known from [6], it is necessary to consider the amplitude of the incident wave as tending toward zero, and the reflection coefficients toward infinity, so that the amplitudes of the reflected waves shall remain finite. Thus, we shall obtain a wave process propagating along the boundary without an incident wave, i.e., we shall have the case of a surface wave. According to (2.6), the reflection coefficients become infinite under the condition

$$(F_1 - L_{12}F_2)\cos 2\theta_t + \mu(\sin 2\theta_1 - L_{12}\sin 2\theta_2)\sin 2\theta_t = 0 \quad (3.1)$$

Let us at first disregard viscosity and therefore consider all coefficients to be real. We insert the symbols

$$S = \sin^2 \theta_t, \quad q_1 = \left(\frac{c_t}{c_1} \right)^2, \quad q_2 = \left(\frac{c_t}{c_2} \right)^2 \quad (3.2)$$

Employing the relationship $\frac{\sin \theta_1}{c_1} = \frac{\sin \theta_2}{c_2} = \frac{\sin \theta_t}{c_t}$, we reduce equation (3.1) to the form

$$(1 - 2S) \left[K_1 - 2\mu \frac{S}{q_1} - L_{12} \left(K_2 - 2\mu \frac{S}{q_2} \right) \right] + 4\mu S \left(\frac{1}{q_1} \sqrt{q_1 - S} - \frac{L_{12}}{q_2} \sqrt{q_2 - S} \right) \sqrt{1 - S} = 0 \quad (3.3)$$

Let us assume conditions (2.13). For a surface wave to exist, it is necessary that equation (3.3) have the root $S > 1$. Hence, the sines of angles θ_1 , θ_2 , and θ_t will be greater than unity, and the cosines purely imaginary, and, consequently, the surface wave will decay exponentially with distance from the boundary. Along the boundary, it will propagate without attenuation at a phase velocity

$$v = \frac{c_t}{\sqrt{S}} \quad (3.4)$$

At $S \rightarrow \infty$ the left-hand member of equation (3.3) reduces to the form

$$- 2S \left[K_1 - \mu - L_{12} (K_2 - \mu) \right] \quad (3.5)$$

and at $S = 1$ it is equal to

$$- \left[K_1 - \frac{2\mu}{q_1} - L_{12} \left(K_2 - \frac{2\mu}{q_2} \right) \right] \quad (3.6)$$

Thus, equation (3.3) has at least one root $s > 1$, if expressions (3.5) and (3.6) are of opposite sign. If we take into account viscosity, the corresponding root of equation (3.6) will be complex, and, consequently, the phase velocity (3.4) also will be complex. This indicates that attenuation of the surface wave takes place in a direction parallel to the boundary, owing to the dissipation of its energy due to viscosity.

In the limiting case of a disappearing viscosity, (3.3) develops into the well-known equation describing the propagation of a surface wave at the boundary between a continuous elastic medium and a vacuum [6].

$$(1 - 2S)^2 + 4S \sqrt{q_1 - S} \sqrt{1 - S} = 0 \quad (3.7)$$

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